AERO 632: Design of Advance Flight Control System

Preliminaries

Raktim Bhattacharya

Laboratory For Uncertainty Quantification
Aerospace Engineering, Texas A&M University.
Preliminaries

- Signals & Systems
- Laplace transforms
- Transfer functions – from ordinary linear differential equations
- System interconnections
- Block diagram algebra – simplification of interconnections
- General feedback control system interconnection.
Signals & Systems
Actuator applies $u(t)$
Sensor provides $y(t)$

Feedback controller takes $y(t)$ and determines $u(t)$ to achieve desired behavior
The controller is typically implemented as software, running in a micro controller

Imperfections exist in real world
- sensors have noise
- actuators have irregularities
- plant $P$ is not fully known
System Response to $u(t)$

Given plant $P$ and input $u(t)$, what is $y(t)$?

- $P$ is defined in terms of ordinary differential equations
- $y(t)$ is the forced + initial condition response.

**Linear Dynamics**

$m\ddot{x} + c\dot{x} + kx = u(t)$ dynamics

$y(t) = x(t)$ measurement

**Nonlinear Dynamics**

$\ddot{x} - \mu(1 - x^2)\dot{x} + x = u(t)$ dynamics

$y(t) = x(t)$ measurement

In this class we focus on linear systems
Linear Systems

Dynamics is defined by linear ordinary differential equation

- Super position principle applies

\[
\begin{align*}
u_1(t) &\leftrightarrow y_1(t) \\
u_2(t) &\leftrightarrow y_2(t) \\
(u_1(t) + u_2(t)) &\leftrightarrow (y_1(t) + y_2(t))
\end{align*}
\]
Laplace Transforms
Laplace Transforms

Given signal $u(t)$, Laplace transform is defined as

$$\mathcal{L}\{u(t)\} := \int_0^\infty u(t)e^{-st}dt$$

Exists when

$$\lim_{t \to \infty} |u(t)e^{-\sigma t}| = 0, \text{ for some } \sigma > 0$$

Very useful in studying linear dynamical systems and designing controllers
Properties Laplace Transforms

Linear operator

- **Additive**

\[
\mathcal{L} \{ u_1(t) + u_2(t) \} = \int_0^\infty (u_1(t) + u_2(t)) e^{-st} \, dt
\]

\[
= \int_0^\infty u_1(t) e^{-st} \, dt + \int_0^\infty u_2(t) e^{-st} \, dt
\]

\[
= \mathcal{L} \{ u_1(t) \} + \mathcal{L} \{ u_2(t) \}
\]

- **Superposition**

\[
\mathcal{L} \{ au(t) \} = a \mathcal{L} \{ u(t) \}, \quad a \text{ is a constant}
\]
Properties (contd.)

1. \( U(s) := \mathcal{L} \{ u(t) \} \)
2. \( \mathcal{L} \{ au_1(t) + bu_2(t) \} = a \mathcal{L} \{ u_1(t) \} + b \mathcal{L} \{ u_2(t) \} = aU_1(s) + bU_2(s) \)
3. \( \frac{1}{s} U(s) \iff \int_0^t u(\tau) d\tau \)
4. \( U_1(s)U_2(s) \iff u_1(t) \ast u_2(t) \) Convolution
5. \( \lim_{s \to 0} sU(s) \iff \lim_{t \to \infty} u(t) \) Final value theorem
6. \( \lim_{s \to \infty} sU(s) \iff u(0^+) \) Initial value theorem
7. \( -\frac{dU(s)}{ds} \iff tu(t) \)
8. \( \mathcal{L} \left\{ \frac{du}{dt} \right\} \iff sU(s) - su(0) \)
9. \( \mathcal{L} \{ \ddot{u} \} \iff s^2U(s) - su(0) - \dot{u}(0) \)
Important Signals

1. \[ \mathcal{L} \{ \delta(t) \} = 1 \] \(\delta(t)\) is impulse function

2. \[ \mathcal{L} \{ 1(t) \} = \frac{1}{s} \] \(1(t)\) is unit step function at \(t = 0\)

3. \[ \mathcal{L} \{ t \} = \frac{1}{s^2} \]

4. \[ \mathcal{L} \{ \sin(\omega t) \} = \frac{\omega}{s^2 + \omega^2} \]

5. \[ \mathcal{L} \{ \cos(\omega t) \} = \frac{s}{s^2 + \omega^2} \]
Transfer Functions
Spring Mass Damper System

Equation of Motion

\[ m\ddot{x} + c\dot{x} + kx = u(t) \]

Take \( \mathcal{L}\{\cdot\} \) on both sides

\[
\mathcal{L}\{m\ddot{x} + c\dot{x} + kx\} = \mathcal{L}\{u(t)\} \\
m\mathcal{L}\{\ddot{x}\} + c\mathcal{L}\{\dot{x}\} + k\mathcal{L}\{x\} = \mathcal{L}\{u(t)\} \\
m(s^2X(s) - sx(0) - \dot{x}(0)) + c(sX(s) - x(0)) + kX(s) = U(s) \\
(ms^2 + cs + k)X(s) = U(s) \]

\( \dot{x}(0) \) and \( x(0) \) are assumed to be zero
Transfer Function

\[(ms^2 + cs + k)X(s) = U(s) \implies \frac{X(s)}{U(s)} = \frac{1}{ms^2 + cs + k}\]

Choose output \(y(t) = x(t) \implies Y(s) = X(s)\).

Therefore

\[P(s) := \frac{Y(s)}{U(s)} = \frac{1}{ms^2 + cs + k}\]
Transfer Function (contd.)

In general

\[ P(s) = \frac{N(s)}{D(s)} \]

where \( N(s) \) and \( D(s) \) are polynomials in \( s \)

- Roots of \( N(s) \) are the **zeros**
- Roots of \( D(s) \) are the **poles** – determine stability
Response to $u(t)$

Given
- input signal $u(t)$ and transfer function $P(s)$.

Determine
- output response $y(t)$

1. **Laplace transform**
   
   $$U(s) := \mathcal{L}\{u(t)\}$$

2. **Determine** $Y(s) := P(s)U(s)$

3. **Laplace inverse**

   $$y(t) := \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\{P(s)U(s)\}$$
System Interconnection
Block Diagram

*Representation of System Interconnections*

- Series
- Parallel
- Feedback
- A simple example
- A complex example
Series Connection

\[ u \rightarrow G_1 \rightarrow G_2 \rightarrow y \]
Parallel Connection

\[ G_1 \quad G_2 \]

\[ u \rightarrow G_1 \rightarrow + \rightarrow G_2 \rightarrow y \]
Feedback Connection

\[ r \rightarrow + \rightarrow G \rightarrow - \rightarrow y \]

AERO 632, Instructor: Raktim Bhattacharya
Simple Example

\[ r \rightarrow G_1 \rightarrow G_2 \rightarrow G_3 \rightarrow \cdots \rightarrow y \]
Complex Example

\[ r \to G_1 \to G_2 \to G_3 \to y \]

\[ + \]

\[ - \]

\[ + \]
Frequency Response
Response to Sinusoidal Input

Let \( u(t) = A_u \sin(\omega t) \)

Vary \( \omega \) from 0 to \( \infty \)

A linear system’s response to sinusoidal inputs is called the system’s frequency response
Response to Sinusoidal Input

Example

■ Let \( P(s) = \frac{1}{s+1} \), \( u(t) = \sin(t) \)

\[
y(t) = \frac{1}{2} e^{-t} - \frac{1}{2} \cos(t) + \frac{1}{2} \sin(t)
\]

\[
= \frac{1}{2} e^{-t} + \frac{1}{\sqrt{2}} \sin\left(t - \frac{\pi}{4}\right)
\]

natural response

forced response

■ Forced response has form \( A_y \sin(\omega t + \phi) \)

■ \( A_y \) and \( \phi \) are functions of \( \omega \)
Response to Sinusoidal Input

Generalization

In general

\[ Y(s) = G(s) \frac{\omega_0}{s^2 + \omega_0^2} \]

\[ = \frac{\alpha_1}{s - p_1} + \ldots + \frac{\alpha_n}{s - p_n} + \frac{\alpha_0}{s + j\omega_0} + \frac{\alpha_0^*}{s - j\omega_0} \]

\[ \implies y(t) = \alpha_1 e^{p_1 t} + \ldots + \alpha_n e^{p_n t} + A_y \sin(\omega_0 + \phi) \]

\[
\begin{align*}
\text{natural} & \quad \text{forced}
\end{align*}
\]

Forced response has same frequency, different amplitude and phase.
Response to Sinusoidal Input

Generalization (contd.)

For a system $P(s)$ and input

$$u(t) = A_u \sin(\omega_0 t),$$

forced response is

$$y(t) = A_u M \sin(\omega_0 t + \phi),$$

where

$$M(\omega_0) = |P(s)|_{s=j\omega_0} = |P(j\omega_0)|,$$

magnitude

$$\phi(\omega_0) = \angle P(j\omega_0),$$

phase

In polar form

$$P(j\omega_0) = Me^{j\phi}.$$
Fourier Analysis
Fourier Series Expansion

Given a signal $y(t)$ with periodicity $T$,

$$y(t) = \frac{a_0}{2} + \sum_{n=1,2,...} a_n \cos \left( \frac{2\pi nt}{T} \right) + b_n \sin \left( \frac{2\pi nt}{T} \right)$$

$$a_0 = \frac{2}{T} \int_0^T y(t) dt$$

$$a_n = \frac{2}{T} \int_0^T y(t) \cos \left( \frac{2\pi nt}{T} \right) dt$$

$$b_n = \frac{2}{T} \int_0^T y(t) \sin \left( \frac{2\pi nt}{T} \right) dt$$
Fourier Series Expansion

Approximation of step function
Fourier Transform

Step function

Fourier transform reveals the frequency content of a signal
Fourier Transform

Step function – frequency content
Signals & Systems

Input Output
\[ u(t) \rightarrow P \rightarrow y(t) \]

Fourier Series Expansion
superposition principle
\[ \sum_i u_i(t) \rightarrow P \rightarrow \sum_i y_i(t) \]

Fourier Transform
\[ U(j\omega) \rightarrow P \rightarrow Y(j\omega) \]

\[ u_i(t) = a_i \sin(\omega_i t) \]
\[ y_{i_{\text{forced}}}(t) = a_i M \sin(\omega_i t + \phi) \]
\[ Y(j\omega) = P(j\omega)U(j\omega) \]

Suffices to study \( P(j\omega), |P(j\omega)|, \angle P(j\omega) \)
Bode Plot
**First Order System**

- \( P(s) = \frac{1}{s + 1} \)
- loglog scale
- \( dB = 10 \log_{10}(\cdot) \)
- \( 20 dB = 10 \log_{10}(100/1) \)

\[ u(t) = A \sin(\omega_0 t) \]

\[ y_{\text{forced}}(t) = AM \sin(\omega_0 t + \phi) \]
Second Order System

- \( P(s) = \frac{1}{s^2 + 0.5s + 1} \)
- \( \omega_n = 1 \text{ rad/s} \)

- \( u(t) = A \sin(\omega_0 t) \)
- \( y_{\text{forced}}(t) = AM \sin(\omega_0 t + \phi) \)
\( S(j\omega) + T(j\omega) = 1 \)

\[
\begin{align*}
|S(j\omega)| & = 10^{-2} \quad \text{for} \quad \omega = 0 \\
|T(j\omega)| & = 10^{-1} \quad \text{for} \quad \omega = 0 \\
|S(j\omega)| & = 10^0 \quad \text{for} \quad \omega = 0 \\
|T(j\omega)| & = 10^1 \quad \text{for} \quad \omega = 0
\end{align*}
\]

\[ \begin{array}{c}
P(s) = \frac{1}{(s+1)(s/2+1)} \\
C(s) = 10 \\
S = G_{er} = \frac{1}{1+PC} = \frac{1}{1+10P} \\
T = G_{yr} = \frac{PC}{1+PC} = \frac{10P}{1+10P}
\end{array} \]
All transfer functions

With proportional controller

\[
G_{er}, \quad G_{ed}, \quad G_{en}, \quad G_{yr}, \quad G_{yd}, \quad G_{yn}
\]
Controller Design Considerations
Design Using Bode Plot of $P(j\omega)C(j\omega)$

Loop Shaping

Develop conditions on the Bode plot of the open loop transfer function

- Sensitivity $\frac{1}{1+PC}$
- Steady-state errors: slope and magnitude at $\lim_{\omega \to 0}$
- Robust to sensor noise
- Disturbance rejection
- Controller roll off $\implies$ not excite high-frequency modes of plant
- Robust to plant uncertainty

Look at Bode plot of $L(j\omega) := P(j\omega)C(j\omega)$
Frequency Domain Specifications

Constraints on the shape of $L(j\omega)$

- Choose $C(j\omega)$ to ensure $|L(j\omega)|$ does not violate the constraints
- Slope $\approx -1$ at $\omega_c$ ensures $PM \approx 90^\circ$
  
  Stable if $PM > 0 \implies \angle PC > -180^\circ$
Plant Uncertainty

\[ P(j\omega) = P_0(j\omega)(1 + \Delta P(j\omega)) \]
Sensor Characteristics

Noise spectrum

\[ G_{yn} = - \frac{PC}{1 + PC} \]
Reference Tracking

**Bandlimited** else conflicts with noise rejection

\[ G_{yr} = \frac{PC}{1 + PC} \]

\[ G_{yn} = -\frac{PC}{1 + PC} \]
Disturbance Rejection

Bandlimited else conflicts with noise rejection

\[
G_{yd} = \frac{P}{1 + PC}
\]

\[
G_{yn} = -\frac{PC}{1 + PC}
\]