Frobenius-Perron Operator
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Frobenius-Perron Operator

Linear Operator $\mathcal{P}_t$

Given dynamics

$$\dot{x} = F(t, x),$$

with $p(t_0, x)$ as the initial state density function.

- Evolution of density is given by

$$p(t, x) := \mathcal{P}_t p(t_0, x).$$

- $\mathcal{P}_t$ has following properties

  $$\mathcal{P}_t (\lambda_1 p_1 + \lambda_2 p_2) = \lambda_1 \mathcal{P}_t p_1 + \lambda_2 \mathcal{P}_t p_2$$  linearity

  $$\mathcal{P}_t p \geq 0 \text{ if } p \geq 0,$$  positivity

  $$\int_{\mathcal{X}} \mathcal{P}_t p(t_0, x) \mu(dx) = \int_{\mathcal{X}} p(t_0, x) \mu(dx)$$  measure preserving

$\mathcal{P}_t$ is defined by

$$\frac{\partial p}{\partial t} + \nabla \cdot (p F) = 0$$

- Continuity equation
- FPK without diffusion term
- First order linear PDE
First Order PDEs

Method of Characteristics

\[
\frac{\partial p}{\partial t} + \nabla \cdot (p F) = \frac{\partial p}{\partial t} + \sum_{i=1}^{n} \frac{\partial p F_i(t, \mathbf{x})}{\partial x_i}
\]

\[
= \frac{\partial p}{\partial t} + \sum_{i=1}^{n} \frac{\partial p}{\partial x_i} F_i(t, \mathbf{x}) + p \sum_{i=1}^{n} \frac{\partial F_i(t, \mathbf{x})}{\partial x_i} = 0
\]

This is of the form

\[
a(t, \mathbf{x}, p) \frac{\partial p}{\partial t} + \sum_{i} b_i(t, \mathbf{x}, p) \frac{\partial p}{\partial x_i} = c(t, \mathbf{x}, p).
\]

Lagrange-Charpit equations

\[
\frac{dt}{a(t, \mathbf{x}, p)} = \frac{dx_i}{b_i(t, \mathbf{x}, p)} = \frac{dp}{c(t, \mathbf{x}, p)}
\]
Characteristic Equations

Lagrange-Charpit equations

\[
\frac{dt}{a(t, x, p)} = \frac{dx_i}{b_i(t, x, p)} = \frac{dp}{c(t, x, p)}
\]

- Let \( s \) be parameterization of characteristic curves
- Characteristic curves are given by the ODEs

\[
\frac{dt}{ds} = a(t, x, p)
\]

\[
\frac{dx_i}{ds} = b_i(t, x, p)
\]

\[
\frac{dp}{ds} = c(t, x, p)
\]
Solution of Continuity Equation

For continuity equation

\[
\frac{\partial p}{\partial t} + \sum_{i=1}^{n} \frac{\partial p}{\partial x_i} F_i(t, \mathbf{x}) + p \sum_{i=1}^{n} \frac{\partial F_i(t, \mathbf{x})}{\partial x_i} = 0
\]

\[
a(t, \mathbf{x}, p) = 1, \quad b_i(t, \mathbf{x}, p) = F_i(t, \mathbf{x}), \quad c(t, \mathbf{x}, p) = -p \sum_{i=1}^{n} \frac{\partial F_i(t, \mathbf{x})}{\partial x_i}.
\]

Characteristic equations

\[
\frac{dt}{ds} = 1 \quad \frac{d\mathbf{x}_i}{ds} = F_i(t, \mathbf{x}) \quad \frac{dp}{ds} = -p \sum_{i=1}^{n} \frac{\partial F_i(t, \mathbf{x})}{\partial x_i}
\]

\[
\dot{\mathbf{x}} = F(t, \mathbf{x}) \quad \text{evolution of } x(t) \\
\dot{p} = -p(\nabla \cdot F) \quad \text{evolution of } p \text{ along } x(t)
\]

Initial Conditions

- \( \mathbf{x}_0 \) Samples from \( p(t_0, \mathbf{x}) \)
- \( p_0 = p(t_0, \mathbf{x}_0) \) Values of \( p(t_0, \mathbf{x}) \) at \( \mathbf{x}_0 \)
Parametric Uncertainty & Process Noise

Given system dynamics

\[ \dot{x} = F(t, x, \Delta) + n(t, \omega) \]

- Expand \( n(t, \omega) \) using KL expansion.
- New parameters: \( \xi := (\xi_0, \xi^*_0, \cdots, \xi_N, \xi^*_N)^T \)
- PDF: \( p_\xi(\xi) \)
- Parameter PDF: \( p_\Delta(\Delta) \)
- State IC PDF: \( p_x(t_0, x) \)

Augment state space

\[ X := \begin{pmatrix} x \\ \Delta \\ \xi \end{pmatrix}, \quad \text{with} \quad \dot{X} := \begin{pmatrix} G(t, x, \Delta, \xi) \\ 0 \\ 0 \end{pmatrix} = H(t, X) \]

with \( p_X(t_0, X) := p_x(t_0, x)p_\Delta(\Delta)p_\xi(\xi) \) and \( p_X(t, X) := \mathcal{P}_t p_X(t_0, X) \).
Better Accuracy & Faster Convergence than MC

(a) First Moment
(b) Second Moment

- Data generated from univariate normal distribution
- MC: PDF from kernel density estimation
- FP: PDF from spline interpolation
- Samples generated 1000 times for a given size. Plots show average error vs sample size

Requires $\frac{\partial F_i(x)}{\partial x_i}$. 
Nonlinear Example

3 DOF Vinh’s Equation Models motion of spacecraft during planetary entry

\[
\begin{align*}
\dot{h} &= V \sin(\gamma) \\
\dot{V} &= -\frac{\rho R_0}{2B_c} V^2 - \frac{g R_0}{v_c^2} \sin(\gamma) \\
\dot{\gamma} &= \frac{\rho R_0}{2B_c} \frac{C_L}{C_D} V + \frac{g R_0}{v_c^2} \cos(\gamma) \left( \frac{V}{R_0 + h} - \frac{1}{V} \right).
\end{align*}
\]

- \(R_0\) – radius of Mars
- \(\rho\) – atmospheric density
- \(v_c\) – escape velocity
- \(\frac{C_L}{C_D}\) – lift over drag
- \(B_c\) – ballistic coefficient
- \(h\) – height
- \(V\) – velocity
- \(\gamma\) – flight path angle
3DOF Vinh's Equation

- Gaussian initial condition uncertainty in \((h, V, \gamma)\)
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Papers

